A topological definition of the Maslov bundle

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Abstract

We give a definition of the Maslov fibre bundle for a lagrangian submanifold of the cotangent bundle of a smooth manifold. This definition generalizes the definition given, in homotopic terms, by Arnol'd for lagrangian submanifolds of $T^*\mathbb{R}^n$. We show that our definition coincides with the one of Hörmander in his works about Fourier Integral Operators.
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Contents

1 Introduction

The Maslov index appears as the phase term when one tries to define the symbol of a Fourier Integral Operator (FIO). This symbol is then defined as a section of the Maslov bundle contructed on a lagrangian submanifold of T^*X. In his historical paper [7], Hörmander proposes a construction of this bundle in terms of cocycles and tries to make the links with the strictly topological presentation (representation of the fundamental group) proposed by Arnol'd [3], originally in an appendix of the book of Maslov [12]. This link is established only for the lagrangian submanifolds of $T^*\mathbb{R}^n$. I propose in this work a new construction (1.2) for the lagrangian submanifolds of T^*X, X a smooth manifold, based on a definition of the Maslov index (1.1) which generalize the one of Arnol'd, and satisfies the cocycles conditions of Hörmander. These correspondances are established in the sections 2 and 3.

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1.1 Arnol'd's definition of the Maslov index

Recall first the construction of Arnol'd [3]. The space $T^*\mathbb{R}^n$ has a symplectic structure by the standard symplectic form

$$\omega = \sum_{j=1}^{j=n} d\xi_j \wedge dx_j.$$

Let $\mathbb{L}(n)$ be the Grassmannian manifold of the Lagrangian subspaces of $T^*\mathbb{R}^n$; we identify $\mathbb{L}(n) = U(n)/O(n)$. The map Det^2 is well defined on $\mathbb{L}(n)$. it is showed in [3] that every path $\gamma : \mathbb{S}^1 \to \mathbb{L}(n)$ such that $Det^2 \circ \gamma \; : \mathbb{S}^1 \to \mathbb{S}^1$ is a generator of $\Pi_1(\mathbb{S}^1)$, gives a generator of $\Pi_1(\mathbb{L}(n))$. It follows that $\Pi_1(\mathbb{L}(n)) \simeq \mathbb{Z}$ and that the cocycle μ_0 defined by

$$\forall \gamma \in \Pi_1(\mathbb{L}(n)) \quad \mu_0(\gamma) = \text{Degree } (Det^2 \circ \gamma)$$

is a generator of the group $H^1(\mathbb{L}(n)) \simeq \mathbb{Z}$. It is then possible to define a *Maslov bundle* $\mathbb{M}(n)$ on $\mathbb{L}(n)$ by the representation $\exp(i\frac{\pi}{2}\mu_0) = i^{\mu_0}$ of $\Pi_1(\mathbb{L}(n))$. It is a flat bundle with torsion because $\mathbb{M}(n)^{\otimes 4}$ is trivial.

Now the Maslov bundle of a submanifold $\mathcal{L}$ of $T^*\mathbb{R}^n$ is the pullback of $\mathbb{M}(n)$ by the natural map

$$\begin{array}{rcl} \varphi_n : \mathcal{L} & \to & \mathbb{L}(n) \\ \nu & \mapsto & T_\nu\mathcal{L}. \end{array}$$

Arnol'd precisely shows that $\mu = {\varphi_n}^*\mu_0$ is the Maslov index of $\mathcal{L}$. One can write

$$\begin{array}{rcl} \mu : \Pi_1(\mathcal{L}) & \to & \mathbb{Z} \\ [\gamma] & \mapsto & < \mu_0, \varphi_n \circ \gamma >= \text{Degree } (Det^2 \circ \varphi_n \circ \gamma). \end{array} \tag{1.1.1}$$

We have to take care of the structural group of this bundle. As a $U(1)-$bundle it is always trivial. But it is concidered as a $\mathbb{Z}_4 = \{1, i, -1, -i\}$-bundle. In fact one can see, using the expression of the Maslov cocycle σ_{jk} given by [7] (3.2.15) that the Chern classes of this bundle are null but σ_{jk} can not be writen in general as the coboundary of a *constant* cochain.

We recall now the theorem of symplectic reduction as it is presented in [6] Proposition 3.2. p.132 .

Proposition 1.1 (Guillemin, Sternberg) . — *Let Δ be an isotropic subspace of dimension m in $T^*\mathbb{R}^{(n+m)}$. Define $S_\Delta = \{\lambda \in \mathbb{L}(n+m)/ \; \lambda \supset \Delta\}$. Then S_Δ is a submanifold of $\mathbb{L}(n+m)$ of codimension $(n+m)$, if we define ρ to be the map*

$$\begin{array}{rcl} \mathbb{L}(n+m) & \xrightarrow{\rho} & \mathbb{L}(n) \\ \lambda & \mapsto & \lambda \cap \Delta^\omega / \lambda \cap \Delta \end{array}$$

(Δ^ω is the orthogonal of Δ for the canonical symplectic form ω), then the map ρ, which is continue on the all $\mathbb{L}(n+m)$, is smooth in restriction to $\mathbb{L}(n+m) - S_\Delta$ and defines on this space a fibre structure with base $\mathbb{L}(n)$ and fibre $\mathbb{R}^{(n+m)}$.

Moreover the image by ρ of the generator of $\Pi_1(\mathbb{L}(n+m))$ is a generator of $\Pi_1(\mathbb{L}(n))$.

1.2 Hörmander's definition of the Maslov bundle

Let X be a smooth manifold, then $T^*X \overset{\pi_0}{\to} X$ is endowed with a canonical symplectic structure by $\omega = d\xi \wedge dx$. Let $\mathcal{L}$ be a lagrangian (homogeneous) submanifold of T^*X. Hörmander, in [7] p.155, defines the Maslov bundle of $\mathcal{L}$ by its sections.

A Lagrangian manifold owns an atlas such that the cards (C_ϕ, D_ϕ) are defined by non degenerated phase functions ϕ defined on $U \times \mathbb{R}^N$ U open in a domain diffeomorphic to a ball of a card of X and

$$\begin{array}{rcl} C_\phi = \Big\{(x,\theta);\ \phi'_\theta(x,\theta) = 0\Big\} & \overset{D_\phi}{\longrightarrow} & \mathcal{L}_\phi \subset \mathcal{L} \\ (x,\theta) & \longmapsto & (x, \phi'_x(x,\theta)). \end{array}$$

For the function ϕ, to be non degenerate means that ϕ'_θ is a submersion and thus C_ϕ is a submanifold and D_ϕ an immersion.
A section is then given by a family of functions

$$z_\phi : C_\phi \to \mathbb{C}$$

satisfying the change of cards formulae :

$$z_{\tilde{\phi}} = \exp i\frac{\pi}{4}\Big(\mathrm{sgn}\phi''_{\theta\theta} - \mathrm{sgn}\tilde{\phi}''_{\tilde{\theta}\tilde{\theta}}\Big) z_\phi. \tag{1.2.2}$$

In fact $(\mathrm{sgn}\phi''_{\theta\theta} - \mathrm{sgn}\tilde{\phi}''_{\tilde{\theta}\tilde{\theta}})$ is even (see below, proposition 3.4) and we have indeed constructed by this way a $\mathbb{Z}_4-$bundle.

1.3 Definition of the Maslov index and results

In the same situation as before, we can construct on any lagrangian submanifold $\mathcal{L}$ of T^*X (and in fact on all T^*X) the following fibre bundle

$$\begin{array}{ccc} \mathbb{L}(n) & \overset{i}{\longrightarrow} & \mathbb{L}(\mathcal{L}) \\ & & \pi\Big\downarrow \\ & & \mathcal{L} \end{array}$$

of the lagrangian subspaces of $T_\nu(T^*X), \nu \in \mathcal{L}$.

This bundle has two natural sections :

$$\lambda(\nu) = T_\nu(\mathcal{L}),\ \text{and}\ \lambda_0(\nu) = \mathrm{vert}(T_\nu(T^*X))$$

defined by the tangent to $\mathcal{L}$ and the tangent to the vertical $T^*_{\pi_0(\nu)}X$.

To a fibre bundle is associated a long exact sequence of homotopy groups, here :

$$...\Pi_2(\mathcal{L}) \to \Pi_1(\mathbb{L}(n)) \overset{i_*}{\to} \Pi_1(\mathbb{L}(\mathcal{L})) \overset{\pi_*}{\to} \Pi_1(\mathcal{L}) \to \Pi_0(\mathbb{L}(n)) = 0.$$

But our fibre bundle possesses a section (two in fact), as a consequence the maps $\Pi_k(\mathbb{L}(\mathcal{L})) \overset{\pi_*}{\to} \Pi_k(\mathcal{L})$ are onto and the maps $\Pi_{k+1}(\mathcal{L}) \to \Pi_k(\mathbb{L}(n))$ are null ; this gives a split exact sequence

$$0 \to \Pi_1(\mathbb{L}(n)) \overset{i_*}{\to} \Pi_1(\mathbb{L}(\mathcal{L})) \overset{\pi_*}{\to} \Pi_1(\mathcal{L}) \to 0.$$

Take a base point $\nu_0 \in \mathcal{L}$ and fix a path σ from $\lambda(\nu_0)$ to $\lambda_0(\nu_0)$ lying in the fibre $\mathbb{L}(\mathcal{L})_{\nu_0}$. For $\gamma \in \Pi_1(\mathcal{L})$ we denote ${\lambda_0}^\sigma{}_*(\gamma)$ the composition of σ, $\lambda_{0*}\gamma$ and finaly σ^{-1} (we use here the conventions of writing of [11]).

Then $\forall \gamma \in \Pi_1(\mathcal{L}),\ \pi_*\Big(\lambda_*\gamma * ({\lambda_0}^\sigma{}_*(\gamma^{-1}))\Big) = 0$ and $\lambda_*\gamma * ({\lambda_0}^\sigma{}_*(\gamma^{-1}))$ is in $\Pi_1(\mathbb{L}(n))$. Let us take the

Definition 1.1 . — *The Maslov index of $\mathcal{L}$ is the map μ :*

$$\forall \gamma \in \Pi_1(\mathcal{L}),\ \mu(\gamma) = \mu_0\Big(\lambda_* \gamma * {\lambda_0}^{\sigma}{}_*(\gamma^{-1})\Big).$$

Proposition 1.2 . — *This definition does not depend on the path σ that we have chosen to joint $\lambda(\nu_0)$ to $\lambda_0(\nu_0)$; moreover μ is a morphism of group, that is : $\mu \in H^1(\mathcal{L}, \mathbb{Z})$.*

First remark : in the case where $X = \mathbb{R}^n$ the fibre bundle $\mathbb{L}(\mathcal{L})$ can be trivialized in such a way that the section λ_0 is constant. In this case our definition coincide with the one of [3]. A natural consequence of the proposition is the following definition :

Definition 1.2 . — *The Maslov bundle $\mathbb{M}(\mathcal{L})$ over $\mathcal{L}$ is defined as in section 1.1 by the representation* $\exp(i\frac{\pi}{2}\mu) = i^{\mu}$ *of $\Pi_1(\mathcal{L})$ in $\mathbb{C}$.*

This means that the sections of the bundle are identified with functions f on the universal cover of $\mathcal{L}$ with compex values and satisfying the relation :

$$\forall \gamma \in \Pi_1(\mathcal{L}), \quad f(x.\gamma) = i^{-\mu(\gamma)} f(x), \tag{1.3.3}$$

like in [2] formula (2.19).

Theorem 1.1 . — *The sections of the Maslov bundle of a Lagrangian (homogeneous) submanifold as defined by the definition 1.2 satisfy the gluing conditions of Hörmander, it means that our definition coincides with the one of Hörmander.*

2 Study of the index μ.

2.1 The index μ_0 on $\mathbb{L}(n)$ is also an intersection number.

For $\alpha \in \mathbb{L}(n)$ et $k \in \mathbb{N}$ one defines $\mathbb{L}^k(n)(\alpha) = \{\beta \in \mathbb{L}(n);\ \dim \alpha \cap \beta = k\}$. Since [3] we know that $\mathbb{L}^k(n)(\alpha)$ is an open submanifold of codimension $\frac{k(k+1)}{2}$, in particular $\overline{\mathbb{L}^1(n)(\alpha)}$ is an oriented cycle of codimension 1 and his intersection number coincides with μ_0.

2.2 Proof of the proposition 1.2.

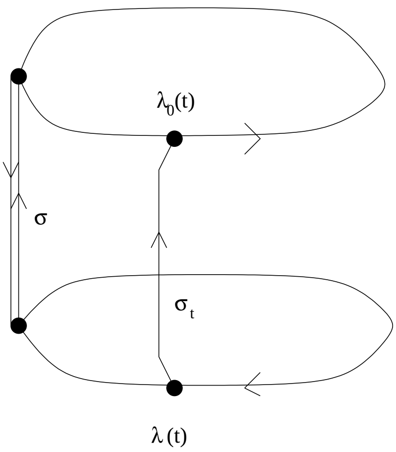
It is a consequence of the two following lemmas. Provide $\mathbb{L}(\mathcal{L})$ with a connection of $U(n)$-bundle. Indeed any symplectic manifold (M, ω), like T^*X, can be provided with an almost complex structure J which is compatible with the symplectic structure(see [1] p.102), it means such that $g(X, Y) = \omega(JX, Y)$ is a riemannian metric. By this way the tangent bundle of M is provided with an hermitian form $g_{\mathbb{C}} = g + i\omega$, and its structural group restricts to $U(n)$ it is also the case for the grassmannian of Lagrangians or its restriction to a submanifold.

We will denote by $\tau(\gamma)_{x\to y}$ the parallel transport for this connection from $\mathbb{L}(\mathcal{L})_x$ to $\mathbb{L}(\mathcal{L})_y$ along the path γ joining x to y in $\mathcal{L}$.

Let's now $\gamma :\ \mathbb{S}^1 \to \mathcal{L}$ be a closed path such that $\gamma(0) = \nu_0$, we define $\lambda(t) = \lambda_*(\gamma)(t)$ and in the same way ${\lambda_0}^{-1}(t) = \lambda_{0*}(\gamma^{-1})(t)$.

If, as before, σ is a path from $\lambda(0)$ to $\lambda_0(0)$ in the fibre $\mathbb{L}(\mathcal{L})_{\gamma(0)}$; then the path of $\mathbb{L}(\mathcal{L})$: $\lambda * \sigma * {\lambda_0}^{-1} * \sigma^{-1}$ is homotopic to a path in the fibre, we have to calculate the Maslov index μ_0 of this last one. For this we use the parallel transport along γ to deform $\lambda * \sigma * {\lambda_0}^{-1}$.

Definition 2.1 . — *For $t \in [0,1]$ let's σ_t denote the path included in the fibre $\mathbb{L}(\mathcal{L})_{\gamma(t)}$ joining $\lambda(t)$ to $\lambda_0(t)$ and obtained by the parallel transport of $\lambda_{|[t,1]} * \sigma * (\lambda_{0|[t,1]})^{-1}$.*



This path has three distinct parts : first $\tilde{\lambda}(t,s) = \tau(\gamma^{-1})_{\gamma(s)\to\gamma(t)}\lambda(s)$ then $\tilde{\sigma}(t,s) = \tau(\gamma^{-1})_{\gamma(1)\to\gamma(t)}\sigma(s)$ and finally $\tilde{\lambda}_0^{-1}(t,s) = \tau(\gamma^{-1})_{\gamma(s)\to\gamma(t)}({\lambda_0}^{-1}(t))$.

By the definition (1.2)

$$\mu(\gamma) = \mu_0(\sigma_0 * \sigma^{-1}).$$

Lemma 2.1 . — *This definition does not depend on the path σ chosen to link $\lambda(0)$ to $\lambda_0(0)$ staying in the fibre above $\gamma(0)$.*

The index μ_0 is defined on the free homotopy group so

$$\mu_0(\sigma_0 * \sigma^{-1}) = \mu_0(\sigma^{-1} * \sigma_0) = \mu_0(\sigma^{-1} * \tilde{\lambda} * \tilde{\sigma} * \tilde{\lambda}_0^{-1})$$

if, here, $\tilde{\lambda}(s) = \tilde{\lambda}(0,s)$ and the same notations for λ_0 and σ.

If σ' is an other path from $\lambda(0)$ to $\lambda_0(0)$, then by the preceding remark and the fact that μ_0 is a morphism of group, one has :

$$\begin{aligned}
\mu_0(\sigma'_0 * {\sigma'}^{-1}) - \mu_0(\sigma_0 * \sigma^{-1}) &= \mu_0({\sigma'}^{-1} * \sigma'_0) - \mu_0(\sigma^{-1} * \sigma_0) = \\
\mu_0({\sigma'}^{-1} * \sigma'_0) + \mu_0({\sigma_0}^{-1} * \sigma) &= \mu_0({\sigma'}^{-1} * \sigma'_0 * {\sigma_0}^{-1} * \sigma) = \\
\mu_0({\sigma'}^{-1} * \tilde{\lambda} * \tilde{\sigma}' * \tilde{\lambda}_0^{-1} * (\tilde{\lambda}_0^{-1})^{-1} * \tilde{\sigma}^{-1} * \tilde{\lambda}^{-1} * \sigma) &= \mu_0({\sigma'}^{-1} * \tilde{\lambda} * \tilde{\sigma}' * \tilde{\sigma}^{-1} * \tilde{\lambda}^{-1} * \sigma) = \\
\mu_0(\sigma * {\sigma'}^{-1} * \tilde{\lambda} * \tilde{\sigma}' * \tilde{\sigma}^{-1} * \tilde{\lambda}^{-1}) &= \mu_0\Big((\sigma * {\sigma'}^{-1}) * \tilde{\lambda} * (\tilde{\sigma} * \tilde{\sigma'}^{-1})^{-1} * \tilde{\lambda}^{-1}\Big) = \\
\mu_0(\sigma * {\sigma'}^{-1}) + \mu_0(\tilde{\lambda} * (\tilde{\sigma} * \tilde{\sigma'}^{-1})^{-1} * \tilde{\lambda}^{-1}) &= \\
\mu_0(\sigma * {\sigma'}^{-1}) + \mu_0(\tilde{\lambda}^{-1} * \tilde{\lambda} * (\tilde{\sigma} * \tilde{\sigma'}^{-1})^{-1}) &= \\
\mu_0(\sigma * {\sigma'}^{-1}) + \mu_0((\tilde{\sigma} * \tilde{\sigma'}^{-1})^{-1}) &= \mu_0(\sigma * {\sigma'}^{-1}) - \mu_0(\tilde{\sigma} * \tilde{\sigma'}^{-1}) = 0
\end{aligned}$$

because $\tilde{\sigma} * \tilde{\sigma'}^{-1}$ is the image of $\sigma * {\sigma'}^{-1}$ by the parallel transport $\tau(\gamma)$ along γ ; but $\tau(\gamma) \in U(n)$ preserves the Maslov index μ_0. ∎

Lemma 2.2 . — *μ is a morphism of groups.*

Indeed, if α and $\beta$are two elements of $\Pi_1(\mathcal{L})$ it is suffisant to calculate $\mu(\alpha) + \mu(\beta)$ beginning the first circle at $\tilde{\sigma}^{-1}(1) = \tau(\alpha)\sigma(0)$ and applying $\tau(\alpha)$ to the second circle which was chosen to begin at $\sigma(0)$. ∎

3 Links with the definition of Hörmander

To make the link of this definition with signature terms of the formula in [7] we follow the calculation from [4].

3.1 Maslov's index in term of signature.

Let $\gamma \in \mathbb{L}^k(n)(\alpha)$ and $\beta \in \mathbb{L}^0(n)(\alpha) \cap \mathbb{L}^0(n)(\gamma)$. Then α and β are transversal and γ can be presented as a graph : there exists a unique linear map $C : \alpha \to \beta$ such that $\gamma = \{(x, Cx), x \in \alpha\}$. [4] p. 181,defines a quadratic form in α by :

$$Q(\alpha, \beta; \gamma) = \omega(C., .) \in \mathcal{Q}(\alpha). \tag{3.1.4}$$

One sees easily that $\ker Q(\alpha, \beta; \gamma) = \ker C = \alpha \cap \gamma$. and if we choose a basis on α such that $Q(\alpha, \beta; \gamma)$ has the form $\begin{vmatrix} B_0 & 0 \\ 0 & 0 \end{vmatrix}$, the null part corresponds to $\alpha \cap \gamma$.

Let now $\gamma(t)$ be a path in $\mathbb{L}^0(n)(\beta)$ such that $\gamma(0) = \gamma$. The goal of the following calculations is to control the jump of the signature of the quadratic form $Q(\alpha, \beta; \gamma(t))$ in the neighbourhood of $t = 0$.

Proposition 3.1 . — *Let $\gamma(t)$ be a path in $\mathbb{L}^0(n)(\beta)$ such that $\gamma(0) = \gamma$. If*

$$Q(\alpha, \beta; \gamma(t)) = \begin{vmatrix} B(t) & C(t) \\ C^t(t) & D(t) \end{vmatrix}$$

with $D(t)$ in $\alpha \cap \gamma$. Then, if $D'(t)$ is invertible in the neighbourhood of 0, there exists $\varepsilon > 0$ such that

$$\forall t,\ 0 < t < \varepsilon \ \ \mathrm{sgn\,Q}(\alpha, \beta; \gamma(\mathrm{t})) - \mathrm{sgn\,Q}(\alpha, \beta; \gamma(-\mathrm{t})) = 2\mathrm{sgn\,D}'(0).$$

Proof. — We know that $B(t)$ is invertible and $C(t)$, $D(t)$ are small. The identity

$$\begin{vmatrix} B & C \\ C^t & D \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ C^tB^{-1} & 1 \end{vmatrix} . \begin{vmatrix} B & 0 \\ 0 & (D - C^tB^{-1}C) \end{vmatrix} . \begin{vmatrix} 1 & B^{-1}C \\ 0 & 1 \end{vmatrix} \tag{3.1.5}$$

gives sgn $Q(\alpha, \beta; \gamma(t))$ =sgn$(B(t))$+sgn$(D(t) - C(t)^tB(t)^{-1}C(t))$. When t is small sgn $B(t)$ = sgn $Q(\alpha, \beta; \gamma)$ and sgn$\Big(D(t) - C(t)^tB(t)^{-1}C(t)\Big)$ = sgn(t) sgn$(D'(0))$ by the mean value theorem. ■

Now if γ is a path which cross *transversally* $\mathbb{L}^1(n)(\alpha)$ at $\gamma(0)$ then the assumption on D' is satisfied.

Theorem 3.1 . — *Let $\alpha \in \mathbb{L}(n)$ and γ a closed path in $\mathbb{L}(n)$ which cross $\mathbb{L}^1(n)(\alpha)$ transversally, then for all $\beta \in \mathbb{L}(n)$ transversal to α and to $\gamma(t)$ one has*

$$\mu_0(\gamma) = \frac{1}{2} \sum_{t, \gamma(t) \in \mathbb{L}^1(n)(\alpha)} \Big(\mathrm{sgn\,Q}(\alpha, \beta; \gamma(\mathrm{t}^+)) - \mathrm{sgn\,Q}(\alpha, \beta; \gamma(\mathrm{t}^-))\Big).$$

Indeed, in this case $T_\gamma\mathbb{L}(n)/T_\gamma\mathbb{L}^1(n)(\alpha) \sim S^2(\alpha \cap \gamma)$ which is oriented by the positive-definite quadratic forms and $\mathrm{sgn\,D}'(0) = \pm 1$, we use then the previous formula.

Remark 3.1 . — *This formula allows to define index of path not necessarely closed, see [13].*

3.2 Hörmander's index.

Let α, β, β' be three elements of $\mathbb{L}(n)$ such that $\beta, \beta' \in \mathbb{L}^0(n)(\alpha)$. For any path σ joining β to β' one defines

$$[\sigma, \alpha] = \mu_0(\hat{\sigma})$$

where $\hat{\sigma}$ is the closed path obtained from σ by linking its endpoints staying in $\mathbb{L}^0(n)(\alpha)$:

$$\hat{\sigma} = \sigma * \sigma_\alpha \text{ and } \sigma_\alpha \subset \mathbb{L}^0(n)(\alpha).$$

The theorem (3.1) shows that $[\sigma, \alpha]$ does not depend on the way σ is closed staying in $\mathbb{L}^0(n)(\alpha)$. Let now α' be a point in $\mathbb{L}^0(n)(\beta) \cap \mathbb{L}^0(n)(\beta')$. The *index of Hörmander* is the number

$$s(\alpha, \alpha'; \beta, \beta') = [\sigma, \alpha'] - [\sigma, \alpha] = \mu_0(\sigma * \sigma_{\alpha'} * (\sigma * \sigma_\alpha)^{-1}) = \mu_0(\sigma_{\alpha'} * {\sigma_\alpha}^{-1})$$

because the calculation of μ_0 does not depend on the base point in $\mathbb{S}^1$.

This index depends only on the four points in $\mathbb{L}(n)$ and not on the paths :

Proposition 3.2 . — *Let $\beta,\ \beta' \in \mathbb{L}^0(n)(\alpha) \cap \mathbb{L}^0(n)(\alpha')$ then*

$$s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2}\Big(\operatorname{sgn} \mathrm{Q}(\alpha, \beta'; \alpha') - \operatorname{sgn} \mathrm{Q}(\alpha, \beta; \alpha')\Big).$$

Indeed, first suppose that α and α' are transversal ; the theorem (3.1) can be applied and also the proposition (3.1) ; this gives

$$s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2}\Big(\operatorname{sgn} \mathrm{Q}(\alpha, \alpha'; \beta) - \operatorname{sgn} \mathrm{Q}(\alpha, \alpha'; \beta')\Big).$$

On the other hand $\beta \in \mathbb{L}^0(n)(\alpha)$ can be writen as the graph of $C \in \mathrm{End}(\alpha, \alpha')$ and so $Q(\alpha, \alpha'; \beta) = \omega(C., .)$. But also α' is the graph of $D \in \mathrm{End}(\alpha, \beta)$ with $\forall x \in \alpha, \quad D(x) = -\big(x + C(x)\big)$, then $Q(\alpha, \beta; \alpha') = \omega(D., .) = -\omega(C., .) = -Q(\alpha, \alpha'; \beta)$. As a consequence

$$s(\alpha, \alpha'; \beta, \beta') = \frac{1}{2}\Big(\ \operatorname{sgn}\ Q(\alpha, \beta'; \alpha') -\ \operatorname{sgn}\ Q(\alpha, \beta; \alpha')\Big).$$

This formula can be generalized by the symplectic reduction (1.1). ■

Let us recall finally the

Proposition 3.3 . — *Let $\alpha, \alpha', \beta, \beta'$ be four points in $\mathbb{L}(n)$ such that β and β' are in $\mathbb{L}^0(n)(\alpha) \cap \mathbb{L}^0(n)(\alpha')$ then*

$$s(\alpha, \alpha'; \beta, \beta') = -s(\alpha', \alpha; \beta, \beta') = -s(\alpha, \alpha'; \beta', \beta) = -s(\beta, \beta'; \alpha, \alpha').$$

Only the third equality is not obvious. It can be shown by the formula of proposition 3.2. Choose symplectic coordinates (x, ξ) such that $\alpha = \{x = 0\}$ and $\beta = \{\xi = 0\}$. By the transversality hypothesis there exist homomorphisms A and B such that

$$\alpha' = \{x = A\xi\} \quad \beta' = \{\xi = Bx\}.$$

If α' is the graph of $A' \in \mathrm{Hom}(\alpha, \beta')$, then for all $\xi \in \alpha$ we must find $\xi' \in \alpha$ and $x \in \beta$ with

$$A'\xi = (x, Bx) \text{ and } (A\xi', \xi') = (x, Bx + \xi).$$

This gives $x = A\xi'$ and $\xi' = Bx + \xi = BA\xi' + \xi$ so $\xi' = (1 - BA)^{-1}\xi$ and

$$A'\xi = \big(A(1 - BA)^{-1}\xi, (1 - BA)^{-1}\xi - \xi\big).$$

We remark that $(1 - BA)$ is indeed invertible : if $\xi \in \ker(1 - BA)$ then $(A\xi, \xi) = (A\xi, BA\xi) \in \alpha' \cap \beta' = \{0\}$ so $\xi = 0$.

Therefore by the proposition (3.2)

$$2s(\alpha, \alpha'; \beta, \beta') = \ \operatorname{sgn}\ \omega(A(1 - BA)^{-1}., .) -\ \operatorname{sgn}\ \omega(A., .) = \ \operatorname{sgn} \begin{vmatrix} A & 0 \\ 0 & -A(1 - BA)^{-1} \end{vmatrix}.$$

Suppose now that A is inversible then, because a symmetric matrix and its inverse have same signature :

$$\operatorname{sgn} \begin{vmatrix} A & 0 \\ 0 & -A(1 - BA)^{-1} \end{vmatrix} = \ \operatorname{sgn} \begin{vmatrix} A & 0 \\ 0 & -(1 - BA)A^{-1} \end{vmatrix} =$$

$$= \ \operatorname{sgn} \begin{vmatrix} A & 0 \\ 0 & B - A^{-1} \end{vmatrix} = \ \operatorname{sgn} \begin{vmatrix} A & 1 \\ 1 & B \end{vmatrix}$$

by formula (3.1.5). By the same calculus, and because ω is skewsymmetric, one has :

$$2s(\beta, \beta'; \alpha, \alpha') = \ \operatorname{sgn}\ Q(\beta, \alpha'; \beta') -\ \operatorname{sgn}\ Q(\beta, \alpha; \beta')) = -\ \operatorname{sgn} \begin{vmatrix} B & 1 \\ 1 & A \end{vmatrix}.$$

■

3.3 Proof of theorem 1.1

Following [7], we denote by $\mathcal{T}(\mathcal{L}) \subset \mathbb{L}(\mathcal{L})$ the set of the $\alpha \in \mathbb{L}(\mathcal{L})$ transversal to $\lambda(\pi(\alpha))$ and to $\lambda_0(\pi(\alpha))$. If $p : \mathcal{T}(\mathcal{L}) \to \mathcal{L}$ is the associated projection, then for all $\nu \in \mathcal{L}$

$$p^{-1}(\nu) = \mathbb{L}^0(n)(\lambda(\nu)) \cap \mathbb{L}^0(n)(\lambda_0(\nu)).$$

n.b. On the neighbourhood of points where the two Lagrangian are not transversal this map is not a fibration.

Lemma 3.1 . — *Let $\alpha :\ \mathbb{S}^1 \to \mathcal{T}(\mathcal{L})$ satisfying $p \circ \alpha = \gamma$ and σ be a path as before. The index $[\sigma_t, \alpha(t)]$ is constant in t.*

Indeed the index is a continuous map : let $t_0 \in [0,1]$ and β a path in the fibre over the point $\gamma(t_0)$ and linking $\lambda_0(t_0)$ to $\lambda(t_0)$ staying tranversal to $\alpha(t_0)$; by definition $[\sigma_{t_0}, \alpha(t_0)] = \mu_0(\sigma_{t_0} * \beta)$ but the property of transversality is open : if we denote β_t the path in the fibre over the point $\gamma(t)$ resulting of the parallel transport of $\lambda_{0|[t,t_0]} * \beta * \lambda^{-1}{}_{|[t,t_0]}$, then there exists $\varepsilon > 0$ such that for all $|t - t_0| < \varepsilon$ one has β_t is transversal to $\alpha(t)$. This parallel transport realizes an homotopy, so for all $|t - t_0| < \varepsilon$ one has $\mu_0(\sigma_{t_0} * \beta) = \mu_0(\sigma_t * \tilde{\beta}_t)$. ■

Corollary 3.1 . — *The induced fibrex bundle $p^*\mathbb{M}(\mathcal{L})$ is trivial.*

Proof. — We have to show that for all path $\alpha : \mathbb{S}^1 \to \mathcal{T}(\mathcal{L})$ continuous, if we define $\gamma = p \circ \alpha$, then $\mu(\gamma) = 0$. To this goal take σ as before, a path in the fibre over $\gamma(0)$ linking $\lambda(0)$ to $\lambda_0(0)$. Choose σ transversal to $\alpha(1)$ and do the same constrution as before, then

$$[\sigma, \alpha(1)] = [\sigma_0, \alpha(0)] = 0$$

by the definition of $[\sigma, \alpha(1)]$ and lemma 3.1. But $\alpha(0) = \alpha(1)$ so

$$\mu(\gamma) = \mu_0(\sigma_0 * \sigma^{-1}) = [\sigma_0, \alpha(1)] = 0.$$

■

Corollary 3.2 . — *Let s be a section of the Maslov bundle over $\mathcal{L}$, and $\gamma :\ \mathbb{S}^1 \to \mathcal{L}$ a closed path such that $\gamma(0) = \nu_0 = \pi(\lambda_0)$. Let $\alpha :\ [0,1] \to \mathcal{T}(\mathcal{L})$ be a continuous path satisfying $\gamma = p \circ \alpha$. Then*

$$p^* s(\alpha(1)) = i^{s(\lambda_0(0), \lambda(0); \alpha(1), \alpha(0))} p^* s(\alpha(0)).$$

Proof. — Let σ be a path linking $\lambda(0)$ to λ_0 staying transversal to $\alpha(1)$. By lemma (3.1), $[\sigma_0, \alpha(0)] = [\sigma, \alpha(1)] = 0$ and

$$\mu(\gamma) = \mu_0(\sigma_0 * \sigma^{-1}) = [\sigma_0, \alpha(1)] = [\sigma_0, \alpha(1)] - [\sigma_0, \alpha(0)] = s(\alpha(0), \alpha(1); \lambda(0), \lambda_0(0))$$

and $s(\alpha(0), \alpha(1); \lambda, \lambda_0) = -s(\lambda_0, \lambda; \alpha(1), \alpha(0))$ by the proposition 3.3. Therefore

$$-\mu(\gamma) = s(\lambda_0(0), \lambda(0); \alpha(1), \alpha(0)).$$

This gives the result by the equivalent relation (1.3.3). ■

From these two corollaries one obtains

Corollary 3.3 . — *The sections of $\mathbb{M}(\mathcal{L})$ are identified with functions f on $\mathcal{T}(\mathcal{L})$ satisfying the relation : $\forall \alpha, \tilde{\alpha} \in \mathcal{T}(\mathcal{L})$*

$$p(\alpha) = p(\tilde{\alpha}) \Rightarrow f(\tilde{\alpha}) = i^{s(\lambda_0, \lambda; \tilde{\alpha}, \alpha)} f(\alpha).$$

This result gives the gluing condition of Hörmander, in view of the theorem 3.3.3, [7] and finish the proof of the theorem. For completness we recall this last step.

Proposition 3.4 . — *The functions f on $\mathcal{T}(\mathcal{L})$ which satisfy : $\forall \alpha, \tilde{\alpha} \in \mathcal{T}(\mathcal{L})$*

$$p(\alpha) = p(\tilde{\alpha}) \Rightarrow f(\tilde{\alpha}) = i^{s(\lambda_0, \lambda; \tilde{\alpha}, \alpha)} f(\alpha).$$

are the sections defined by the gluing conditions of the section 1.2.

Proof. — Let ϕ be a non degenerated phase function as in section 1.2 and $\nu_0 = (x_0, \xi_0) = (x_0, \phi'_x(x_0, \theta_0))$ a point in $\mathcal{L}_\phi$. For each $\alpha \in \mathcal{T}(\mathcal{L})$ such that $p(\alpha) = \nu_0$, there exists a function ψ defined on an open set U such that the graph $L_\psi = \{(x, d\psi(x)),\ x \in U\}$ of the differential $d\psi$ intersect transversally $\mathcal{L}_\phi$ at ν_0, one has $\xi_0 = d\psi(x_0)$ and $T_{\nu_0} L_\psi = \alpha$.

Or equivalently one can say : the following quadratic form defined on $\mathbb{R}^{n+N}$ by the matrix

$$Q_\psi = \begin{vmatrix} \phi''_{xx} - \psi''_{xx} & \phi''_{x\theta} \\ \phi''_{\theta x} & \phi''_{\theta\theta} \end{vmatrix} \tag{3.3.6}$$

is non degenerated.

The restriction of this quadratic form to the tangent W of $\mathcal{L}_\phi$ at ν_0 only depends on $\mathcal{L}$ and ψ (and not on ϕ). Indeed ϕ defines a card in which

$$\lambda(\nu_0) = T_{\nu_0}(\mathcal{L}) = \{(X, \phi''_{xx}X + \phi''_{x\theta}A); (X, A) \in \mathbb{R}^{n+N},\ \phi''_{\theta x}X + \phi''_{\theta\theta}A = 0\};$$

if now $(X, A), (X', A')$ define two tangent vectors V and $V' \in \lambda(\nu_0)$

$$\begin{aligned} Q_\psi\Big((X, A), (X', A')\Big) &= \ < X, (\phi''_{xx} - \psi''_{xx})X' + \phi''_{x\theta}A' > \\ < -\psi''_{xx}X, X' > - < -X, \phi''_{xx}X' + \phi''_{x\theta}A' > &= \ Q\Big(\lambda(\nu_0), \alpha; \lambda_0(\nu_0)\Big)(V, V') \end{aligned}$$

by definition (3.1.4). More precisely α is transverse to the two lagrangians $\lambda(\nu_0)$ and $\lambda_0(\nu_0)$ so the vertical $\lambda_0(\nu_0)$ is the graph of an homomorphism A_ψ from $\lambda(\nu_0)$ to $\alpha = T_{\nu_0} L_\psi$:

$$\forall (0, \Xi) \in \lambda_0(\nu_0),\ \exists (X, A) \text{unique such that } \Xi = \phi''_{xx}X + \phi''_{x\theta}A \text{ et } \phi''_{\theta x}X + \phi''_{\theta\theta}A = 0$$

because Q_ψ is non degenerated, and one can write

$$(0, \Xi) = (X, \phi''_{xx}X + \phi''_{x\theta}A) - (X, \psi''_{xx}X),$$

it means that $A_\psi(X, \phi''_{xx}X + \phi''_{x\theta}A) = (-X, -\psi''_{xx}X)$.

We see now that the orthogonal W^{Q_ψ} of W with respect to Q_ψ is $\mathbb{R}^N = \{(0, A)\}$ and that $Q_{\psi|W^{Q_\psi}} = \phi''_{\theta\theta}$. But the lemma 3.2 below gives $\mathrm{sgn}\, \mathrm{Q}_\psi = \mathrm{sgn}\, \mathrm{Q}_{\psi|\mathrm{W}} + \mathrm{sgn}\, \mathrm{Q}_{\psi|\mathrm{W}^{\mathrm{Q}_\psi}}$, so :

$$\mathrm{sgn}\, \mathrm{Q}_\psi = \mathrm{sgn}\, \mathrm{Q}(\lambda(\nu_0), \alpha; \lambda_0(\nu_0)) + \mathrm{sgn}\, \phi''_{\theta\theta}. \tag{3.3.7}$$

Let now z_ϕ be a section in the sens of Hörmander. For any $\alpha \in \mathcal{T}(\mathcal{L}), p(\alpha) = \nu_0$, if ϕ and $\tilde{\phi}$ are two phase functions defining $\mathcal{L}$ in a neighbourhood of ν_0 and if ψ is a function on X satisfying $\alpha = T_{\nu_0} L_\psi$, we denote by Q_ψ and $\tilde{Q}_\psi$ the respective quadratic forms defined by (3.3.6). Put

$$f(\alpha) = \exp(i \frac{\pi}{4} \mathrm{sgn}\, \mathrm{Q}_\psi) \mathrm{z}_\phi(\nu_0).$$

By the relation (3.3.7) one has $\mathrm{sgn}\, \phi''_{\theta\theta} - \mathrm{sgn}\, \tilde{\phi}''_{\tilde{\theta}\tilde{\theta}} = \mathrm{sgn}\, \mathrm{Q}_\psi - \mathrm{sgn}\, \tilde{\mathrm{Q}}_\psi$; the compatibility condition 1.2.2 gives then

$$\exp(i \frac{\pi}{4} \mathrm{sgn}\, \mathrm{Q}_\psi) \mathrm{z}_\phi(\nu_0) = \exp(\mathrm{i} \frac{\pi}{4} \mathrm{sgn}\, \tilde{\mathrm{Q}}_\psi) \mathrm{z}_{\tilde{\phi}}(\nu_0)$$

and the function f is well defined on $\mathcal{T}(\mathcal{L})$. On the other hand if $\tilde{\alpha}$ is an other point in $\mathcal{T}(\mathcal{L})$ such that $p(\tilde{\alpha}) = \nu_0$ and if $\tilde{\psi}$ is an adapted function, then

$$\begin{aligned} f(\tilde{\alpha}) &= \exp(i \frac{\pi}{4} (\mathrm{sgn}\, \tilde{\mathrm{Q}}_\psi - \mathrm{sgn}\, \mathrm{Q}_\psi)) \mathrm{f}(\alpha) \\ &= \exp\left(i \frac{\pi}{4} \Big(\mathrm{sgn}\, \mathrm{Q}(\lambda(\nu_0), \tilde{\alpha}; \lambda_0(\nu_0)) - \mathrm{sgn}\, \mathrm{Q}(\lambda(\nu_0), \alpha; \lambda_0(\nu_0)\Big)\right) \mathrm{f}(\alpha) \\ &= \exp\left(i \frac{\pi}{2} s(\lambda(\nu_0), \lambda_0(\nu_0); \alpha, \tilde{\alpha})\right) f(\alpha) \\ &= \exp\left(i \frac{\pi}{2} s(\lambda_0(\nu_0), \lambda(\nu_0); \tilde{\alpha}, \alpha)\right) f(\alpha) \end{aligned}$$

So it is a section of the Maslov bundle and the theorem 1.1 is proved. ■

Lemma 3.2 . — *Let Q be a non degenerated quadratic form defined on $\mathbb{R}^n$, V be a subspace of $\mathbb{R}^n$ and V^Q its orthogonal for Q, then*

$$\mathrm{sgn\,Q} = \mathrm{sgn\,Q}_{|\mathrm{V}} + \mathrm{sgn\,Q}_{|\mathrm{V}^\mathrm{Q}}.$$

Proof. — This lemma can be showed using an induction on $\dim V \cap V^Q$. If $\dim V \cap V^Q = 0$ there is nothing to do, if not let $v_1, \ldots, v_k$ be a base of $V \cap V^Q$. We complete this base with $v_{k+1}, \ldots, v_p$ to obtain a base of $V + V^Q$. Because Q is non degenerated there exists $w_1 \in \mathbb{R}^n$ such that $Q(v_1, w_1) = 1$, and eventually after a modification with a linear combination of the v_j one can suppose $Q(w_1) = 0$ and $Q(v_j, w_1) = 0$ for $j > 1$. One remarks that the signature of Q in restriction to $\mathbb{R}v_1 \oplus \mathbb{R}w_1$ is zero and applies the induction hypotheses to $(\mathbb{R}v_1 \oplus \mathbb{R}w_1)^Q$. ■

4 Topological comments

Let's have a look to the exact sequence : $0 \to \Pi_1(\mathbb{L}(n)) \xrightarrow{i_*} \Pi_1(\mathbb{L}(\mathcal{L})) \xrightarrow{\pi_*} \Pi_1(\mathcal{L}) \to 0$.

The group $\Pi_1(\mathbb{L}(\mathcal{L}))$ is the semidirect product of $\Pi_1(\mathbb{L}(n))$ and $\Pi_1(\mathcal{L})$. It means that $\Pi_1(\mathcal{L})$ acts on $\Pi_1(\mathbb{L}(n))$ by conjugation. More precisely for all $\gamma \in \Pi_1(\mathcal{L})$ let's define

$$\begin{array}{rcl} \rho_\gamma : \Pi_1(\mathbb{L}(n)) & \to & \Pi_1(\mathbb{L}(n)) \\ \sigma & \mapsto & \lambda_0(\gamma) * i_*(\sigma) * (\lambda_0(\gamma))^{-1} \end{array}$$

Lemma 4.1 *This representation is trivial and $\Pi_1(\mathbb{L}(\mathcal{L}))$ is in fact the direct product of $\Pi_1(\mathbb{L}(n))$ and $\Pi_1(\mathcal{L})$.*

Proof. — As was seen in paragraph 2, the parallel transport along γ defines an homotopy of $\lambda_0(\gamma) * i_*(\sigma) * (\lambda_0(\gamma))^{-1}$ to a path which can be writen $\tilde{\lambda}_0 * \tilde{\sigma} * (\tilde{\lambda}_0)^{-1}$ where $\tilde{\sigma}$ is the image of σ by $\tau(\gamma)$. But

$$\mu_0(\tilde{\lambda}_0 * \tilde{\sigma} * (\tilde{\lambda}_0)^{-1}) = \mu_0((\tilde{\lambda}_0)^{-1} * \tilde{\lambda}_0 * \tilde{\sigma}) = \mu_0(\tilde{\sigma}) = \mu_0(\sigma).$$

As a consequence of the works of Arnol'd recalled above, a generator of $\Pi_1(\mathbb{L}(n))$ is caracterized by $\mu_0(\sigma) = 1$. ■

Theorem 4.1 . — *Let $\mathbb{L}^1(\mathcal{L})$ be the set of the points $l \in \mathcal{L}$ which are not transversal to $\lambda_0(\pi(l))$. It is an oriented cycle of $\mathcal{L}$ of codimension 1 ; if m is its Poincaré dual form, then*

$$\mu = \lambda^* m.$$

Proof. — We keep the notations of paragraph 2. By choosing the starting point one can suppose that *the two lagrangians $\lambda_0 = \lambda_0(0)$ and $\lambda(0)$ are transversal.* We will use a deformation of the path $\tilde{\lambda} * \tilde{\sigma} * \tilde{\lambda}_0^{-1}$ joining $\lambda(0)$ to $\lambda_0(0)$. Recall that $\tilde{\sigma}(t) = \tau(\gamma)(\sigma(t))$.

There exists a (continuous) path $u(t) \in U(n)$ such that $u(0) =$I and

$$\forall t \in [0,1]\ \tilde{\lambda}_0(t) = u(t)(\lambda_0).$$

But $\tilde{\lambda}_0(1) = \tau(\gamma)(\lambda_0)$, so $\tau(\gamma)$ and $u(1)$ differ by an element of $O(n)$:

$$\exists a \in O(n)\ ;\ \tau(\gamma) = u(1) \circ a.$$

Let's construct the following homotopy of $\tilde{\lambda} * \tilde{\sigma} * \tilde{\lambda}_0^{-1}$ by the concatenation of $u(st)^{-1}\tilde{\lambda}(t)$, next $u(s)^{-1}\tilde{\sigma}$ and finally the inverse of $u(st)^{-1}\tilde{\lambda}_0(t)$. The end of this homotopy is a path, result of the concatenation of $\bar{\lambda}(t) = u(t)^{-1}\tilde{\lambda}(t)$ and $u(1)^{-1}\tilde{\sigma} = a\sigma$ because $u(t)^{-1}\tilde{\lambda}_0(t) = \lambda_0$ is a constant path.

We have now to calculate $\mu_0(\sigma^{-1} * \bar{\lambda} * a\sigma)$. Because $a \in O(n)$

$$Det^2(\sigma(t)) = Det^2(a\sigma(t));$$

$Det^2 \circ \bar{\lambda}$ is a closed path even if $\bar{\lambda}$ is not, so $\mu(\gamma) =$ Degree $(Det^2 \circ \bar{\lambda})$.

Considering the results of section 2.1, we have obtained

Proposition 4.1 *$\mu(\gamma)$ is the intersecting number of the submanifold $\overline{\mathbb{L}^1(n)(\lambda_0)}$ and the cycle obtained from $\bar{\lambda}$, by closing it with a path staying transversal to λ_0.*

Remark that $\bar{\lambda}(0) = \lambda(0)$ and $\bar{\lambda}(1) = a\lambda(0)$ are both transversal to λ_0. Let's now

$$\mathbb{L}^1(\mathcal{L}) = \Big\{ l \in \mathbb{L}(\mathcal{L})\ ;\ \lambda_0(\pi(l)) \cap l \neq \{0\} \Big\}.$$

It is a fibration above $\mathcal{L}$ with fibre $\overline{\mathbb{L}^1(n)(\lambda_0)}$, so it is an oriented cycle of codimension 1 in $\mathcal{L}$. If $\lambda \circ \gamma$ cuts $\mathbb{L}^1(\mathcal{L})$ transversally at $\lambda \circ \gamma(t)$ then $\bar{\lambda}$ cuts transversally $\overline{\mathbb{L}^1(n)(\lambda_0)}$ at $\bar{\lambda}(t)$ and conversely. Moreover the transformations which permit to pass from $\lambda \circ \gamma$ to $\bar{\lambda}$ realise a continuous deformation of $\mathbb{L}^1(\mathcal{L})$ to $\overline{\mathbb{L}^1(n)(\lambda_0)}$ above γ. This argument finishes the proof of the theorem 4.1. ∎

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